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NASH EQUILIBRIA ON (UN)STABLE NETWORKS

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In response to a change, individuals may choose to follow the responses of their friends or, alternatively, to change their friends. To model these decisions, consider a game where players choose their behaviors and friendships. In equilibrium, players internalize the need for consensus in forming friendships and choose their optimal strategies on subsets of k players—a form of bounded rationality. The k -player consensual dynamic delivers a probabilistic ranking of a game's equilibria, and via a varying k , facilitates estimation of such games.

Applying the model to adolescents' smoking suggests that: (a) the response of the friendship network to changes in tobacco price amplifies the intended effect of price changes on smoking, (b) racial desegregation of high schools decreases the overall smoking prevalence, (c) peer effect complementarities are substantially stronger between smokers compared to between nonsmokers.

KEYWORDS: Games on endogenous networks, adolescent smoking, multiplicity.

1. INTRODUCTION

IN RESPONSE TO A CHANGE in their environment, individuals may choose to follow the responses of their friends or, alternatively, choose differently *and* change their friends. In the context of evaluating public policies (e.g., an excise tax on tobacco consumption), this latter alternative motivates a shift from questions such as how the friendship network *propagates* changes in individuals' behaviors (e.g., individuals' smoking choices), say, due to a policy intervention, to questions such as how the friendship network *responds* to such changes in individuals' behaviors. This paper studies this shift in perspective from both a theoretical and public policy view.

In order to do so, consider an environment where individuals choose both their behaviors and friendships. While these choices are fundamentally different, their difference is not related to the presence of strategic incentives or instincts for selfish decisions. Rather, choosing a friend presumes a consent (Jackson and Wolinsky (1996)) while choosing behaviors does not (Nash (1950)). The tension between the instinct for selfish choices and the consensual nature of humans' friendships can be prototyped as a game of link and node statuses where the players' decision problem is augmented with a set of *stability constraints*. These constraints reflect that a player internalizes the need for consent in forming links, or in other words, a player may form her links only with those who desire to be her friends.

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A player's observed friendship links and behaviors are likely to compare favorably against her alternatives, that is, are likely to be robust against a set of deviations. The complexity of individuals' decision problem, captured by the number of possible deviations, motivates a family of equilibria indexed by the radius of permissible deviations.¹ For a fixed parameter k , a Nash equilibrium in a k -stable (NE k S) network emerges when no player has a profitable deviation that is permissible by the stability constraints and that involves less than k links. In the proposed model, all NE k S networks are pairwise stable, and for $k = n$, NE k S networks are pairwise-Nash networks (Proposition 2; Jackson (2005) overviews these concepts).

A primitive feature of games of links and behaviors is payoff externalities, which can lead to multiplicity of NE k S networks. The proposed tool to reconcile this multiplicity is a probabilistic ranking—consistent with the NE k S play—of a game's outcomes. This ranking obtains in a random utility re-formulation of an adaptive dynamic based on (the individual's decision problem from) the NE k S play.^{2 3} More specifically, a k -player consensual dynamic (k CD) is a family of adaptive dynamic processes where players sequentially adapt their behaviors and at most $k - 1$ of their links, of course, subject to the stability constraints (Proposition 3). In the presence of random preference shocks, k CDs induce a unique, invariant to k , stationary distribution over the set of all possible outcomes (Theorem 1). Consistently with the NE k S play, each NE k S network is a local mode of this probability distribution (Theorem 3). The larger k is, the faster a k CD approaches the stationary distribution (Theorem 2).

These properties of k CDs facilitate both estimation of and simulation from these games. The model's likelihood is given by the (unique) stationary distribution of the k CD family. This distribution pertains to the Exponential Random Graph Models (Frank and Strauss (1986), Wasserman and Pattison (1996)), for which both direct estimation and simulating from the model with known parameters are infeasible.⁴ Bayesian estimation strategies have utilized an asymptotic algorithm, the so-called double Metropolis–Hastings sampler, that relies on simulations from the model via Markov chains (Murray, Ghahramani, and MacKay (2006), Liang (2010), Mele (2017)). While, for different k s, k CDs have different convergence properties, they have the same stationary distribution (Theorems 1 and 2), which in turn suggests a transparent template for designing these Markov chains with varying k (Algorithm Table I and Proposition 4).⁵

The model is applied to data on smoking behavior, friendship networks, and home environment (parental education background and parental smoking behavior) from the National Longitudinal Study of Adolescent Health. This is a longitudinal study of a nationally representative sample of adolescents in the United States, who were in grades 7–12 during the 1994–1995 school year.⁶

¹In a game of size n , there are $n - 1$ permissible deviations for a player who considers changing the status of a single link, $3 \frac{(n-1)(n-2)}{2}$ for a player who considers changing two of her links, etc.

²See Train (2003, Chapter 2) for an overview of the random utility approach.

³Similar dynamics, although in a different context, are analyzed by the evolutionary game theory and individual learning literatures, for example, Foster and Young (1990), Kandori, Mailath, and Rob (1993), Blume (1993), Jackson and Watts (2001, 2002). In a crude form, the motivation for these dynamics is present in Cournot (1838, Chapter VII) and Nash (1950, Section 9).

⁴An evaluation of the likelihood involves sums with $2^{(n^2+n)/2}$ terms. As n grows, computing such sums quickly becomes prohibitively time expensive, for example, for $n = 10$ the number of terms nears 10^{17} .

⁵Poor convergence properties are associated with local Markov chains, where each update is of size $o(n)$ (Bhamidi, Bresler, and Sly (2011)). Importantly, varying k on the support $\{2, \dots, n - 1\}$ is not anymore a local Markov chain. I thank an anonymous referee for pointing this out.

⁶Details about these data are available in the Online Supplementary Material.

The empirical analysis of adolescent smoking and friendship selection delivers a host of results which are related to the vast empirical research on social interactions and teen risky behaviors. Typically, empirical studies of peer effects either lack friendship data or take the friendship network as exogenously given.⁷ Also, the approaches range from models that directly relate an individual's choices to mean characteristics of his peer groups (e.g., see Powell, Tauras, and Ross (2005) and Ali and Dwyer (2009)) to models with elaborate equilibrium microfoundations (e.g., see Brock and Durlauf (2001, 2007), Krauth (2005), Calvó-Armengol, Patacchini, and Zenou (2009)). In terms of estimates, this paper is the first assessment of how neglecting the response of the social network to policies targeting teen risky behaviors can bias the estimates of both the model's parameters and the predicted policy outcomes.⁸ In terms of the determinants of teen risky behaviors, this paper pioneers the role of the school composition, or more generally the determinants of the social fabric, in shaping teen decisions.⁹

1.1. *Conclusions From the Empirical Analysis*

The model is estimated under various restrictions on the parameter specification and on the data availability. Two observations merit noting at the estimation stage. First, the peer effect complementarities are substantially stronger between smokers compared to those between nonsmokers. Second, lack of network data, which forces the estimation to suppress the local peer effect externalities, substantially biases downwards the estimate of the price coefficient.

The obtained sets of estimates are used to perform numerical experiments under various counterfactual scenarios. The purpose of these experiments is to quantify the response of the friendship network to policies targeting adolescent smoking. A by-product of this analysis is an assessment of the bias in the model's predictions due to lack of network data or due to various misspecifications.

The first experiment addresses the question of whether the response of the friendship network is relevant for policies working through changes in tobacco prices. To motivate this question, compare how individuals respond to a price increase in fixed versus endogenous network environments. There are two effects to consider. The *direct* effect of changing tobacco prices is the first-order response and, intuitively, is larger in endogenous network environment where individuals are free to change their friendships. That is, more individuals are likely to immediately respond to changes in tobacco prices provided they are not confined to their (smoking) friends. The *indirect* (ripple) effect of changing tobacco prices is the effect on smoking which is due, in part, to the fact that one's friends have stopped smoking. Contrary to before, the indirect effect is larger in a fixed network environment because fixed networks are more likely to propel further the changed behaviors. That is, an individual who changes her smoking status is bound to exert pressure to her (fixed) friends who are most likely smokers. It is then an empirical question how these two opposing effects balance out. Simulations with the full model and with a model where the friendship network is kept fixed suggest that the direct effect dominates. In

⁷See, for example, Liu, Patacchini, and Zenou (2014), who distinguish between local aggregate and local average peer effects, and the references therein.

⁸It is difficult, if not impossible, to account for the empirical contributions of the large literature on peer effects and teen risky behaviors. For a small sample of papers obtaining estimates of peer effects, see Chaloupka and Wechsler (1997), Ali and Dwyer (2009), and the references in CDC (2000, Surgeon General's Report).

⁹The possibility of such a role was theorized by Graham, Imbens, and Ridder (2014) and experimentally discovered by Carrell, Sacerdote, and West (2013).

other words, following an increase in tobacco prices the response of the friendship network amplifies the intended reduction in smoking prevalence.

The second experiment asks whether school racial composition has an effect on adolescent smoking. When students from different racial backgrounds study in the same school, they interact and are likely to become friends. Being from different racial backgrounds, students have different intrinsic propensity to smoke and the question is what is the equilibrium smoking behavior in these mixed-race friendships: do those who do not smoke start smoking or those who smoke stop smoking? Simulations from the model suggest that redistributing students from racially segregated schools into racially balanced schools decreases the overall smoking prevalence.

The last experiment simulates a small scale policy intervention targeting only a part of the school's population. The policy is efficient so that those exposed to the treatment stop smoking. At the same time, it is not feasible (too costly) to treat the entire school. In this experiment, the question is when treated individuals return, will their friends follow their example, that is, extending the effect of the proposed policy beyond the set of treated individuals, and thus creating a domino effect, or will their pre-treatment friends unfriend them? In essence, this is a question about the magnitude of the spillover effects and this study suggests that aggregate spillovers are roughly double compared to the scale of the policy.

1.2. *Related Literature*

This paper studies and estimates a game on endogenous network where players choose both their behaviors (e.g., smoking) and friendship links. The proposed model can be restricted to a game played on a fixed network. These games date back to the physics literature of the 1970s and in economics have been analyzed with both discrete and continuous choices (e.g., see Jackson and Zenou (2015) and Bramoullé and Kranton (2016) for surveys). Most of the empirically tractable games have been developed either in continuous settings (e.g., Ballester, Calvó-Armengol, and Zenou (2006), Bramoullé, Kranton, and D'Amours (2014), Calvó-Armengol, Patacchini, and Zenou (2009)) or when data on the friendship network is not available, restricting the model further to where peer effects are measured via group averages (e.g., see Brock and Durlauf (2001, 2007), Nakajima (2007), and the survey in Blume, Brock, Durlauf, and Jayaraman (2015)).

Symmetrically, the proposed model can be restricted to a network formation game (e.g., see Jackson (2008) for a systematic textbook presentation). A large and growing body of studies on the economics of these games followed Jackson and Wolinsky (1996) who, in a departure from the traditional noncooperative game paradigm, introduced the notion of pairwise stability. In this paper, the stability constraints guarantee that any NE k S play is pairwise stable and for $k = n$ such play is pairwise-Nash (see Myerson (1991), Calvó-Armengol (2004), Goyal and Joshi (2006), Bloch and Jackson (2006, 2007) and the survey in Jackson (2005)).

A handful of theoretical papers consider both network formation along with other choices potentially affected by the network (see Goyal and Vega-Redondo (2005), Cabrales, Calvó-Armengol, and Zenou (2011), König, Tessone, and Zenou (2014), Baetz (2015), Lagerås and Seim (2016), Hiller (2017), Jackson (2018)). Importantly, the theoretical frameworks available are meant to provide focused insights into isolated features of networks and deliver sharp predictions, while abstracting from players' heterogeneity and so are not easily adapted for the purposes of estimation.

Econometric models of networks and actions are proposed in Goldsmith-Pinkham and Imbens (2013), Hsieh and Lee (2016), and Johnsson and Moon (2019) where the decisions to form friendships influence the decision to engage in a particular activity. The focus of their research, however, is not on policy analysis nor on accounting for the possible endogenous response of the friendship network to changing the decision environment. In contrast, the framework proposed in Boucher (2016) is microfounded as a particular equilibrium in a noncooperative model of friendships and behaviors. Related work by Hsieh, König, and Liu (2016) proposes a two-stage estimation procedure, with an application to R and D, which relies on the conditional independence of links obtained from assuming away link externalities. Canen, Trebbi, and Jackson (2016) proposed an empirically tractable framework, building on Cabrales, Calvó-Armengol, and Zenou (2011), where politicians choose both socialization and legislation efforts, and study bill cosponsorship in the U.S. Congress.¹⁰ Different to this literature (including Boucher, Hsieh, and Lee (2019) and Battaglini, Patacchini, and Rainone (2019)), the proposed model is founded on the explicit strategic incentives that guarantee consent and stability in link formation in the sense of Jackson and Wolinsky (1996).

Finally, adaptive dynamic and potential function representation, as a dimensionality reduction tool, are widely used in (algorithmic) game theory, computer science, and in economics of networks for processes on fixed networks, for processes of link formation and, more recently, for combined processes, for examples, Foster and Young (1990), Blume (1993), Jackson and Watts (2001, 2002), Nakajima (2007), Bramoullé, Kranton, and D'Amours (2014), Bourlés, Bramoullé, and Perez-Richet (2017), Mele (2017), Boucher (2016), and Hsieh and Lee (2016). In contrast to this literature, this paper highlights a slightly different role for these tools, namely, to probabilistically rank the equilibria of the static game and to simulate (and estimate) these games.

2. A GAME ON AN ENDOGENOUS NETWORK

Imagine a world where individuals choose both their friends and their behaviors, for example, to smoke or not. A small scale example is depicted as a graph in Figure 1. Individuals are depicted as nodes and the star-shaped shaded nodes are those who smoke.

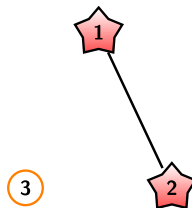


FIGURE 1.—An illustration with 3 individuals (players). *Note:* In the graph, each player in $I = \{1, 2, 3\}$ is depicted as a vertex and a friendship is depicted as an edge, for example, players 1 and 2 are friends. The star-shaped shaded nodes denote players who smoke tobacco, for example, players 1 and 2 are smokers. In the notation from Section 2.1, $a_1 = a_2 = 1$, $a_3 = 0$, $g_{12} = 1$, and $g_{13} = g_{23} = 0$.

¹⁰Recent econometric analyses focus on link formation, though these are not easily extendable to include action choice as well; for example, see Sheng (2020), Chandrasekhar and Jackson (2016), Leung (2015), de Paula, Richards-Shubik, and Tamer (2018), Graham (2017), Menzel (2015), Leung and Moon (2021), and the reviews in Chandrasekhar (2015), de Paula (2016), Bramoullé, Galeotti, and Rogers (2016).

Friendships are depicted as links between pairs of nodes. These links are undirected because (being in) a friendship is a symmetric binary relation.

It is worth pausing to list the defining features of individuals' decision environment. First, individual's choices of behavior and friendships are different in that friendships (unlike behaviors) require consent to form and maintain. This fundamental difference is embedded in the proposed formalization of individual's decision problem. Second, there are likely to be externalities not only between individual's behavior and the behaviors her friends but also between individuals' friendship decisions. These externalities are explicitly specified in players' payoff functions. Finally, this is a complex decision environment in that even with 100 individuals, each one considers roughly 10^{30} alternative strategies. This complexity is reflected in the proposed (family of) equilibria and adaptive dynamics.

The model is developed in two stages. The remainder of Section 2 analyzes agents' strategic choices in the settings of a static game. Section 3 translates agents' static decision problem to the settings of an adaptive dynamic and formally argues that this dynamic delivers an inferentially convenient approximation of the static play.

2.1. Players and Preferences

Each i , in a finite population $I = \{1, 2, \dots, n\}$, chooses $a_i \in \{0, 1\}$ and a set of links $g_{ij} = g_{ji} \in \{0, 1\}$ for $j \neq i$. In the settings of adolescents' smoking and friendship decisions, I is the set of all students in a given high school, $a_i = 1$ if student i smokes, and $g_{ij} = 1$ if i and j are friends. In the illustration from Figure 1 above, there are 3 players so that $I = \{1, 2, 3\}$, and $a_1 = a_2 = 1, a_3 = 0, g_{12} = 1$, and $g_{13} = g_{23} = 0$. A final piece of the description of the population is individuals' exogenous characteristics X_i , for example, age, race, gender, etc.

Player i chooses her behavior and friendships statuses $S_{(i)} = (a_i, \{g_{ij}\}_{j \neq i})$ from her choice set $\mathbf{S}_{(i)} = \{0, 1\}^n$ to maximize her payoff u_i . Let $S = (S_{(1)}, \dots, S_{(n)}) \in \prod_i \mathbf{S}_{(i)} = \mathbf{S}$ and $X = (X_1, \dots, X_n) \in \mathbf{X}$. Formally i 's payoff function, $u_i : \mathbf{S} \times \mathbf{X} \rightarrow \mathbb{R}$, orders the outcomes in \mathbf{S} given X :

$$u_i(S, X) = a_i v_i + \underbrace{a_i \phi \sum_{j \neq i} a_j}_{\text{aggr. externalities}} \tag{1}$$

$$+ \underbrace{\phi_S \sum_j g_{ij} a_i a_j + \phi_N \sum_j g_{ij} (1 - a_i)(1 - a_j)}_{\text{local externalities}} \tag{2}$$

$$+ \underbrace{\sum_j g_{ij} w_{ij} + q_{ijk} \sum_{\substack{j,k \\ j < k}} g_{ij} g_{jk} g_{ki}}_{\text{clustering}} - \underbrace{\psi \left(\frac{1}{2} (d_i^2 + d_i) + \sum_{j \neq i} g_{ij} d_j \right)}_{\text{(convex) cost}_i(d_i, \{d_j\}_{j: g_{ij}=1})} \tag{3}$$

where $d_i = \sum_j g_{ij}$ is the degree (total number of links) of i . Here, $v_i = v(X_i)$, $w_{ij} = w(X_i, X_j)$ and $q_{ijk} = q(X_i, X_j, X_k)$ are functions of agents' (exogenous) characteristics. To avoid clutter in the summation ranges, assume that g_{ii} is defined and equal to zero for all i so that, for example, $d_i = \sum_{j \neq i} g_{ij} = \sum_j g_{ij}$.

The first thing to note about the payoff u_i is the presence of individuals' heterogeneity. The terms in (1)–(3) depend on the exogenous characteristics of i and her friends (e.g.,

terms v_i , w_{ij} and q_{ijk}) and, also, on the endogenous choices of others in the population (e.g., j 's smoking status, degree and presence of common friends). A more subtle point to note, before introducing each payoff term, is that the terms in (1)–(3) can be sorted into three groups: terms that relate to the incremental payoff of changing a_i , terms that relate to the incremental payoff of changing g_{ij} and terms that relate to both.

The first three terms in (1)–(3) relate to the incremental payoff of changing i 's behavior a_i conditional on the friendship network. This incremental payoff is

$$\Delta_{a_i} u_i(S, X) = v_i + \phi \sum_{j \neq i} a_j + \phi_S \sum_{j \neq i} g_{ij} a_j - \phi_N \sum_{j \neq i} g_{ij} (1 - a_j).$$

The first term v_i is the (exogenous) intrinsic utility of choice $a_i = 1$, which is allowed to vary with i 's attributes X_i . The second term $\phi \sum_{j \neq i} a_j$ captures the aggregate externalities. That is, i may be influenced from the behaviors of the surrounding population $\sum_{j \neq i} a_j$, provided $\phi \neq 0$. The last two terms in $\Delta_{a_i} u_i(S, X)$ are the differential of the local externalities $\phi_S \sum_j g_{ij} a_i a_j + \phi_N \sum_j g_{ij} (1 - a_i)(1 - a_j)$ in (2). Note that $a_i a_j$ equals 1 if and only if $a_i = a_j = 1$ so that, conditional on the friendship network, this term captures pressures on i to follow (or to break away if $\phi_S < 0$) her friends' decision to choose 1 (to smoke). Analogously, $(1 - a_i)(1 - a_j)$ equals 1 if and only if $a_i = a_j = 0$, and this term captures pressures on i to conform to the behaviors of her choosing 0 (nonsmoking) friends. Because ϕ_S need not equal ϕ_N , the opposing conformity pressures from friends who choose 1 and from friends who choose 0 need not be equal in magnitude. Finally, as will become evident shortly, the local externalities terms are related to the incremental payoff of changing g_{ij} where, conditional on individuals' actions, these terms capture a tendency to befriend others playing the same action. To sum up, an agent's utility increases by ϕ_S with every friend who plays the same action if that action is 1, and by ϕ_N with every friend who plays the same action if that action is 0.

The last four terms in (1)–(3) relate to the incremental payoff to i of changing g_{ij} conditional on players' actions. This incremental payoff is

$$\Delta_{g_{ij}} u_i(S, X) = w_{ij} + q_{ijk} \sum_k g_{ik} g_{jk} - \psi(d_i + d_j) + \phi_S a_i a_j + \phi_N (1 - a_i)(1 - a_j).$$

The first term w_{ij} captures the (exogenous) utility of a friendship which may depend on i 's and j 's degree of similarity, that is, same age, race, sex, etc. The next term is the differential of $q_{ijk} \sum_{j < k} g_{ik} g_{jk} g_{ki}$ in (3) which captures link externalities. Mechanically, i may have preferences for whether or not her friends are friends themselves. In particular, i may prefer sharing her friends ($q > 0$), or on the contrary, prefer friendship exclusivity ($q < 0$).¹¹ The third term $-\psi(d_i + d_j)$ is the differential of the convex cost term in (3) which reflects the costs of establishing a friendship between i and j . Properties of the cost term to note are: (i) the more friends i has, the more costly it is for i to establish an additional friendship and (ii) the costs are shared so for i it is more costly to maintain friendships with more popular (high d_j) as opposed to less popular (low d_j) individuals. The last two terms relate to the previously discussed local externalities terms $\phi_S \sum_j g_{ij} a_i a_j + \phi_N \sum_j g_{ij} (1 - a_i)(1 - a_j)$ in (2).

¹¹A compelling interpretation of this term is consistent with the presence of meeting frictions. In particular, meeting and befriending friends of friends can explain the tendency of individuals to form triangles of friendships (e.g., see Jackson and Rogers (2007)). This paper studies relatively small friendship networks so frictions are less likely to play a pronounced role.

2.2. Equilibrium Play

Given a player’s preferences, her observed links and action are likely to compare favorably against her alternatives. However, the number of available alternatives renders players’ decision problem complex. In contemplating the optimality of her play, a player has to consider $2^n - 1$ possible deviations. A natural way to restrict the alternatives under consideration is to presume that players consider only strategies that are close by a candidate equilibrium play, or in the settings above, that players consider strategies that involve changing only few link statuses. A final point concerning the equilibrium is that players are aware that links are formed with consent and subject their choices to this information.

DEFINITION 1: A profile of actions and a network $S^* = (\{a_i^*\}_{i \in I}, \{g_{ij}^*\}_{i \in I, j \in I \setminus i})$ is a Nash equilibrium in a k (-player) stable (NEkS) network, provided $S_{(i)}^* = (a_i^*, \{g_{ij}^*\}_{j \in I \setminus i})$ is a solution of i ’s decision problem on $I_k \subseteq I$:

$$\max_{a_i, \{g_{ij}\}_{j \in I_k \setminus i}} u_i(a_i, \{g_{ij}\}_{j \in I \setminus i}; S_{-i}^*) \tag{4}$$

$$\text{s.t.} \quad g_{ij} = 1 \quad \text{only if} \quad \Delta_{g_{ij}} u_j(a_i, \{g_{ij}\}_{j \in I \setminus i}; S_{-i}^*) \geq 0 \quad \forall j \in I_k \setminus i, \tag{5}$$

where $1 < k \leq n, I_k = \{i\} \cup \{i_1, \dots, i_{k-1}\}$ and $i \notin \{i_1, \dots, i_{k-1}\}$, for all i and I_k .

To state the above definition in words, in a NEkS network no player has permissible, by the stability constraints (5), and profitable deviation involving changing the statuses of less than k links. A notable feature of the NEkS networks is that not only links are formed with consent but also players internalize the need for consent through subjecting their play to the stability constraints. The stability constraints owe their name to their relation to the notion of stability introduced in Jackson and Wolinsky (1996) (see Proposition 2 below).

ASSUMPTION 1: Assume that $w()$ and q are symmetric in their arguments/indices.

PROPOSITION 1: With Assumption 1:

1. For any S, k, i , and I_k , the problem in (4)–(5) is well-defined and has a solution;
2. For any k , a NEkS network exists.

The existence of a (nontrivial) solution to individual’s decision problem (4)–(5) and an equilibrium follows from the existence of a potential function for this game (Monderer and Shapley (1996)). The proof is in the Appendix (p. 1199).

PROPOSITION 2: With Assumption 1:

1. For $k = 2$, NEkS networks are pairwise stable;
2. For $k = n$, NEkS networks are pairwise-Nash networks;
3. For $k' < k$, any NEkS network is also a NEk’S network.

Part 1 can be strengthened for any preferences: for $k = 2$, any NEkS play is pairwise stable (Jackson and Wolinsky (1996)). For $k = n$, NEkS networks are pairwise-Nash networks (Calvó-Armengol (2004), Goyal and Joshi (2006), Bloch and Jackson (2006, 2007)). Finally, the NEkS family is ordered by set inclusion so that the existence of a pairwise stable network is a necessary condition for the existence of a NEkS network for $k > 2$. The proof is in the Appendix (p. 1201).



FIGURE 2.—Examples of NE k S networks for $k = 2$ and $k = 3$. Note: Let $I = \{1, 2, 3\}$, $\phi = q = w_{ij} = 0$ and $u_i = v_i + \phi_0 \sum_j g_{ij}(a_i a_j + (1 - a_i)(1 - a_j)) - \psi \text{cost}_i(d_i, \{d_j\}_{j \in d_i})$. It is straightforward to find parameter values for v_i , ϕ_0 and ψ where for $k = 3$ there is a unique NE k S network while for $k = 2$ both of the depicted networks are NE k S networks.

2.3. An Example of NE k S Networks With 3 Players

To see how the choice of k may affect the equilibrium networks, consider a simplified version of payoffs (1)–(3) where all externalities other than the local peer effects and costs are absent. Let $I = \{1, 2, 3\}$, $\phi = 0$, $\phi_S = \phi_N = \phi_0 > 0$, $q = 0$, $\psi > 0$, and $w_{ij} = 0$ for all i and j so that

$$u_i = v_i + \phi_0 \sum_j g_{ij}(a_i a_j + (1 - a_i)(1 - a_j)) - \psi \text{cost}_i(d_i, \{d_j\}_{j \in d_i}). \tag{6}$$

Suppose that: (i) $v_2 = \bar{v}$ and $v_3 = -\bar{v}$ for \bar{v} large so that players 2 and 3 always choose $a_2 = 1$ (smoke) and $a_3 = 0$ (not smoke), respectively, and (ii) the benefits of having a friend that plays the same action outweigh the costs ($\phi_0 > 2\psi$). With these assumptions, player 1 will always choose to befriend either player 2 or player 3 depending on 1’s smoking choice.

The candidates for equilibrium are depicted in Figure 2. For $k = 3$, there is (generically) a unique NE k S network. If $v_1 < 0$, then player 1 chooses not smoke and befriends player 2 (Figure 2 left) else ($v_1 > 0$) player 1 chooses to smoke and befriends player 3 (Figure 2 right). In contrast, for $k = 2$ if $v_1 \in (-\phi_0 + 2\psi, \phi_0 - 2\psi)$ both networks in Figure 2 are NE k S networks. Note that the larger the complementarities are (ϕ_0), the larger the region for v_1 is where there are multiple NE k S networks.

3. CONSENSUAL DYNAMIC. AN ESTIMABLE FRAMEWORK

The NE k S play offers an intuitive prescription for the *outcomes* of the forces driving behaviors and friendships, without specifying the decision *process* leading to these outcomes. This abstraction is challenged by strong informational assumptions where players are presumed to correctly anticipate other players’ choices and, also, by the presence of multiple NE k S networks none of which can be ruled out a priori. Turning to a framework based on an adaptive dynamic (and random utility) delivers a way to embed this multiplicity into an inferentially convenient framework.

Formulation (4)–(5) of individuals’ decision problem provides a basis for a simple dynamic where behaviors and friendships evolve in a way consistent with the NE k S play. The general idea that equilibrium might arise from a simple (myopic) adaptive dynamic as opposed to from a complex reasoning process is very intuitive. In comparison with the interpretations in Kandori, Mailath, and Rob (1993), Blume (1993), and Jackson and Watts (2001, 2002), the emphasis is on obtaining an empirically tractable framework via

a flexible dynamic process (parametrized via k).¹² More specifically, this dynamic process serves a dual purpose. As a conceptual model of individuals' behavior, it delivers the likelihood of a statistical model and justifies the NEkS networks as a highly probable outcome. As an algorithm template, it facilitates the estimation of and simulation from these games because of its particular properties.

3.1. k -Player Consensual Dynamic (kCD)

Every period $t = 1, 2, \dots$ a randomly chosen individual, say i , considers her behavior a_i and $k - 1$ of her friendships, say with $\{i_1, \dots, i_{k-1}\}$, to (myopically) solve her decision problem (4)–(5) on $I_k = \{i\} \cup \{i_1, \dots, i_{k-1}\}$. A stochastic meeting process μ_t outputs i and I_k :

$$\Pr(\mu_t = (i, I_k) | S_{t-1}, X) = \mu_{i, I_k}(S_{t-1}, X). \tag{7}$$

In the simplest case, any meeting is equally probable and $\mu_{i, I_k}(S_{t-1}, X) = \frac{1}{n} \frac{1}{\binom{n-1}{k-1}}$ for all i, I_k, S_{t-1} , and X . Rather, it is only necessary that any meeting is possible.

ASSUMPTION 2: $\mu_{i, I_k}(S_{t-1}, X) > 0$ for all $i \in I, I_k, S \in \mathbf{S}$, and $X \in \mathbf{X}$.

The sequence of random meetings together with players' optimal decisions induce a sequence of network states (S_t) referred to as a k (-player) consensual dynamic (kCD).

PROPOSITION 3: Fix $k \in \{2, \dots, n\}$. With Assumptions 1 and 2, for a kCD S_t :

1. A NEkS network is absorbing, that is, $S_{t'} = S_t$ if $t' > t$ and S_t is a NEkS network;
2. Independently of the initial state $\Pr(\lim_{t \rightarrow \infty} S_t \in \text{NEkSN}) = 1$.

Indeed, for any k , the NEkS networks are exactly the rest points of simple adaptive processes, the kCD s. The proof is in the Appendix (p. 1201).

3.2. kCD s With Random Utility

Consider a modification of a kCD where players' decision problem (4)–(5) is cast as a random utility choice.¹³ More specifically, conditional on the realized meeting i and I_k , player i 's payoffs for each alternative on I_k are augmented with a random component, ultimately making the solution of (4)–(5) stochastic. Such a kCD with random utility delivers a (stationary) distribution over all outcomes as opposed to a single outcome. This distribution has convenient properties when treated as the likelihood function of a statistical model.

ASSUMPTION 3: Suppose that the players' payoffs are given by $u_i(S, X) + \epsilon_S$ where u_i is defined as before in (1)–(3) and ϵ_S is an additive preference shock i.i.d. across time and S . Moreover, suppose that ϵ_S has c.d.f. and unbounded support on \mathbb{R} .

ASSUMPTION 4: Suppose that ϵ_S has a Gumbel($\mu_\epsilon, \beta_\epsilon$) distribution.

¹²Similar dynamics, although in a different context, are studied by the literature on stochastic stability. There, either agents take turns to update their strategies (i.e., in our settings, all links) or a pair of players update the status of their link.

¹³This is also known as the random utility model. See Thurstone (1927), Marschak (1960), McFadden (1974), and for textbook treatment, Train (2003, Chapter 2).

ASSUMPTION 5: Suppose that the meeting probabilities $\mu_{i,I_k}(S, X)$ do not depend on a_i and g_{ij} for all $j \in I_k$. (Alternatively, suppose that $\mu_{i,I_k}(S, X) = \mu_{i,I_k}(S', X)$ for all $S, S' \in \mathbf{S}$, which is a slightly weaker condition but less intuitive.)

The sequence of meetings together with players' (stochastic) choices induce a Markov chain on \mathbf{S} referred to as a k CD with random utility.

THEOREM 1—Stationary Distribution: Fix $k \in \{2, \dots, n\}$. A k CD with random utility has the following properties:

1. With Assumptions 2 and 3, there is a unique stationary distribution $\pi_k \in \Delta(\mathbf{S})$ for which $\lim_{t \rightarrow \infty} \Pr(S_t = S) = \pi_k(S)$. In addition, for any function $f : \mathbf{S} \rightarrow \mathbb{R}$, $\frac{1}{T} \sum_{t=0}^T f(S_t) \rightarrow \int f(S) d\pi_k$.
2. With Assumptions 1–5,

$$\pi(S, X) \propto \exp\left(\frac{\mathcal{P}(S, X)}{\underline{\beta}}\right). \tag{8}$$

In particular, $\pi(S, X)$ does not depend on k .

The first part is not surprising in that it asserts that a k CD with random utility is well behaved so that standard convergence results apply. The uniqueness of π_k precludes dependence between snapshots from this process and its initial state, and the ergodicity allows one to simulate from π_k via drawing a long trajectory of a k CD.

The second part has implications for implementing the model. The stationary distribution π in (8) does not depend on k , and thus, delivers a tool to unify the equilibria in the NE k S family. More specifically, π ranks in a probabilistic sense the family of equilibria within and across different k s (see Theorem 3). This result bears two immediate consequences for implementing the model, when π is treated as the likelihood: (a) the multiplicity of NE k S networks is reconciled and (b) the estimation does not need data on k . A final implication from part two of the theorem is that a closed-form expression for π provides for a transparent identification of the model's parameters¹⁴ and facilitates the use of likelihood-based methods for estimation.

3.3. Speed of Convergence

The k CDs with random utility depend on k despite the invariance of their stationary distribution to k . Below is a formal statement of a theorem on this dependence and a discussion of the theorem's hypothesis, which isolates away all other determinants of the k CDs with random utility.

THEOREM 2— k CDs Ranking: Suppose that $u_i(S, X) = 0$ for all $i, S \in \mathbf{S}$ and $X \in \mathbf{X}$. Then the second eigen value of the $2^{(n^2+n)/2}$ -by- $2^{(n^2+n)/2}$ transition matrix of a k CD with random utility is given by

$$\lambda_{k,[2]} = \frac{1}{n} \left(n - 1 + \frac{n - k}{n - 1} \right). \tag{9}$$

In particular, $\lambda_{k',[2]} < \lambda_{k,[2]}$ for $2 \leq k < k' \leq n$ so that the k' CD with random utility converges strictly faster than k CD with random utility.

¹⁴For more details on identification, see Section 4.4.

The hypothesis of Theorem 2 presumes that players do not differentiate between different networks or, equivalently, that all payoff parameters in (1)–(3) are set to equal 0. To put it another way, the k CDs with random utility traverse in unbiased way the space of all possible networks \mathbf{S} . As a result, in the stationary distribution π the behaviors and network links are i.i.d. Poisson(0.5) and, importantly, k is the only determinant of k CDs' transition probabilities and convergence rates.¹⁵

There are two rationales behind pursuing a characterization of the speed of convergence of k CDs with random utility. As anticipated (and formally established shortly) π probabilistically ranks the family of NE k S networks. In a dual fashion, the differential speed of convergence provides a means to rank the family of k CDs with random utility. In particular, the larger k is, the smaller is the second eigenvalue $\lambda_{k, [2]}$, that is, the faster k CDs converge to π (see [Debreu and Herstein \(1953, Section 4\)](#)).

The second reason for why properties of k CDs are of their own interest is highlighted by [Bhamidi, Bresler, and Sly \(2011\)](#) who show that adaptive dynamic with local updates (i.e., $o(n)$ links at a time) converges very slowly. Such slow convergence rates could question the conceptual treatment of the limiting distribution π as a likelihood. For this same reason, simulation based methods that rely on local updates may not work in practice for estimation/simulation of these models.¹⁶ Note that k CDs encompass not only local updates, for example, $k = \lceil n/2 \rceil$, and thus suggest a way to avoid the problem of slow convergence (poor approximation). Relatedly, Theorem 2 offers insights into an important trade-off for sampling design: the Markov chain is facing a trade-off between speed of convergence and complexity in simulating the next step. For a smaller k , the convergence to π is slower, however, generating an update is faster because this update is drawn from a discrete distribution with smaller (2^k) support.¹⁷

3.4. Discussion

3.4.1. Probabilistic Ranking. The Most Probable Equilibria

The stationary distribution π from Theorem 1 provides an intuitive probabilistic ranking of the family of NE k S networks. In fact, π assigns a positive probability to all possible outcomes of the game including those that are not equilibria. Consistently with a NE k S play, the equilibrium has the highest probability among the outcomes in the neighborhood of that equilibrium. Notably, different NE k S networks are assigned different probabilities and the mode of π , that is, the most probable equilibria, has a special role (see Theorem 3 below). It is worth pointing out that this approach of working with all equilibria via an equilibrium ranking is a departure from the theoretical literature on equilibrium selection from evolutionary game theory.

The neighborhood of an outcome $S \in \mathbf{S}$ is given by the set of all outcomes that differ by the status of a single link or the action from S :

$$\mathbf{N}(S) = \{(g_{ij}, S_{-ij}) : i \neq j, g_{ij} \in \{0, 1\}\} \cup \{(a_i, S_{-i}) : a_i \in \{0, 1\}\}.$$

¹⁵In general, the shape of the potential, that is, the terms of the potential function, and the geography of the network X will likely influence the speed of convergence. To the best of my knowledge, treatment of the general case remains out of reach.

¹⁶See the discussion in [Chandrasekhar and Jackson \(2016\)](#).

¹⁷Relatedly, there is an important computational shortcut when simulating from a k CD. Within the MH algorithm for generating the update of a k CD, computing the acceptance probability scales only quadratically with the size of the network because it is enough to compute the change in utility as opposed to the potential itself. The replication code of the paper contains further details.

THEOREM 3: *Suppose Assumptions 1–5 hold.*

1. *A state $S \in \mathbf{S}$ is a Nash equilibrium in a pairwise stable network iff it receives the highest probability in its neighborhood \mathbf{N} .*
2. *The most likely network states $S^{\text{mode}} \in \mathbf{S}$ (the ones where the network spends most of its time) are pairwise Nash networks.*

3.4.2. Random k

Consider what appears to be a very unrestrictive meeting process, where every period a random individual meets a set of potential friends of random size and composition. Let k be a discrete process with support $\{2, \dots, n\}$ and augment the meeting process with an additional initialization step with respect to the size k . At each period, first k is realized and then I_k is drawn just as before. It is straightforward to establish, without additional assumptions on the process k , that this random k - k CD with random utility has the same stationary distribution π as the one from Theorem 1.¹⁸ This is yet another demonstration that the model is agnostic to the meeting patterns (which are typically not observed).

4. DATA AND ESTIMATION

4.1. The Add Health Data

The National Longitudinal Study of Adolescent Health contains data on a sample of adolescents in grades 7–12 in the United States in the 1994–1995 school year. The sample is representative of US schools with respect to region of country, urbanicity, school size, school type, and ethnicity. In total, 80 high schools were selected together with their “feeder” schools. The students were first surveyed in-school and then at home in four follow-up waves conducted in 1994–1995, 1996, 2001–2002, and 2007–2008. This paper makes use of Wave I of the in-home interviews with students enrolled in the schools from the so-called saturated sample. Only for schools from the saturated sample, all of their students were eligible for in-home interviews.

The in-home interviews contain rich data on students’ behaviors, home environment, and friendship networks. These data are merged with administrative data on the average price of a carton of cigarettes from the American Chamber of Commerce Research Association (ACCRA). ACCRA’s data are linked to the Add Health data on the basis of state and county FIPS codes for the year in which the data were collected. Additional details about the estimation sample including sample construction and sample statistics are presented in the Online Supplementary Material (Badev (2021)).

4.2. Bayesian Estimation

The k CDs with random utility deliver a unique stationary distribution π which can be thought of as the likelihood of a statistical model. Because no information is available on when the process started or on its initial state, the best prediction about the current state is given by π . For a single observation $S \in \mathbf{S}$, the likelihood is¹⁹

$$p(S|\theta) = \frac{\exp\{\mathcal{P}_\theta(S)\}}{H_\theta}, \tag{10}$$

¹⁸A formal statement and a proof are omitted because these follow the ones of Theorem 1.

¹⁹In the expression for π from Theorem 1, the coefficient $\underline{\beta}$ is normalized to equal 1 because \mathcal{P} is linear in the utility parameters so that $\underline{\beta}$ cannot be separately identified.

TABLE I
VARYING k DOUBLE M-H ALGORITHM

Input: initial $\theta^{(0)}$, number of iterations T , size of the Monte Carlo R , data S

1. **for** $t = 1 \dots T$
2. Propose $\theta' \sim q(\theta'; \theta^{(t-1)}, S)$
3. Initialize $S^{(0)} = S$
4. **for** $r = 1 \dots R$
5. Draw $k \sim p_k(k)$
6. Draw a meeting $I_k = \{i\} \cup \{i_1, \dots, i_{k-1}\} \sim \mu_k$
7. Propose S' where $(a_i, \{g_{ij}\}_{j \in I_k})$ are drawn from $q_\mu(S'|S^{(r-1)}; (i, I_k))$
8. Compute $\bar{a} = \frac{\exp\{\mathcal{P}_{\theta'}(S')\}}{\exp\{\mathcal{P}_{\theta'}(S^{(r-1)})\}} \frac{Q(S^{(r-1)}|S'; p_k, q_{i, I_k})}{Q(S'|S^{(r-1)}; p_k, q_{i, I_k})}$
9. Draw $a \sim \text{Uniform}[0, 1]$
10. If $a < \bar{a}$ then $S^{(r)} = S'$ else $S^{(r)} = S^{(r-1)}$
11. **end for** [r]
12. Compute $\bar{a} = \frac{q(\theta^{(t-1)}; \theta')}{q(\theta'; \theta^{(t-1)})} \frac{p(\theta')}{p(\theta^{(t-1)})} \frac{\exp\{\mathcal{P}_{\theta^{(t-1)}}(S^{(R)})\}}{\exp\{\mathcal{P}_{\theta^{(t-1)}}(S)\}} \frac{\exp\{\mathcal{P}_{\theta'}(S)\}}{\exp\{\mathcal{P}_{\theta'}(S^{(R)})\}}$
13. Draw $a \sim \text{Uniform}[0, 1]$
14. If $a < \bar{a}$ then $\theta^{(t)} = \theta'$ else $\theta^{(t)} = \theta^{(t-1)}$
15. **end for** [t]

where \mathcal{P}_θ is the potential (evaluated at θ) and $H_\theta = \sum_{S \in \mathcal{S}} \exp\{S\}$ is an (intractable) normalizing constant. In practice, H_θ cannot be computed directly even for small n , for example, for $n = 10$ this summation includes $2^{55} \approx 10^{17}$ terms.²⁰

The estimation draws from the Bayesian literature on approximating intractable likelihoods (Murray, Ghahramani, and MacKay (2006), Liang (2010), Mele (2017)), modified with sampling from a random k - k CDs with random utility. The posterior sampling algorithm is exhibited in Table I. In the original double M-H algorithm, an M-H sampling of S from $\pi_\theta(S)$ is nested in an M-H sampling of θ from the posterior $p(\theta|S)$. The new piece in Table I is the random k in step 5. Theorem 2 suggests that varying k improves the convergence while Theorem 1 demonstrates that this leaves the stationary distribution unchanged. The validity of the proposed modification is demonstrated below.

PROPOSITION 4—Varying k Double M-H Algorithm: *Let $1 < k \leq n$ and suppose Assumptions 1–5 hold. If in the algorithm of Table I, the proposal density conditional on meeting (i, I_k) , $q_\mu(S'|S; (i, I_k))$ is symmetric, then the unconditional proposal $Q(S'|S)$ is symmetric. In particular, the acceptance probability of the inner M-H step does not depend on p_k or q_μ .*

The Bayesian estimator requires specifying prior distributions and proposal densities. All priors $p(\theta)$ are normal and all proposals (p_k , μ , and q_μ) are uniform over their respective domains.

4.3. Parametrization

The payoffs from (1) and (2) have six sets of parameters: v_i , w_{ij} , q , ϕ , ϕ_S , and ϕ_N . In the empirical specification, the first three are functions of the data $v_i = V(X_i)$,

²⁰The specific form of the likelihood pertains to the exponential family, whose application to graphical models has been termed as Exponential Random Graph Models (ERGM). See Frank and Strauss (1986) and Wasserman and Pattison (1996).

$w_{ij} = W(X_i, X_j)$, $q_{ijk} = q(X_i, X_j, X_k)$. Careful scrutiny of the data and extensive experimentation with various parametrizations motivate the final specification, which is discussed in the Online Supplementary Material (Badev (2021)).

4.4. Identification

Because the model pertains to the exponential family, identification within the framework of many networks follows immediately. Indeed, a corollary of Theorem 1 is that the likelihood of the model is proportional to $\exp\{\sum_{r=1}^R \theta_r w_r(S, X)\}$, where $w_r : \mathbf{S} \times \mathbf{X} \rightarrow \mathbb{R}$ are functions of the data. To obtain identification, it is enough that the sufficient statistics w_r are linearly independent functions on $\mathbf{S} \times \mathbf{X}$ (e.g., see Lehmann and Casella (1998) for a textbook treatment). In the structural model, this condition is readily established.²¹

Unobservable Heterogeneity

In addition to the models’ parameters for observable attributes, it is possible to incorporate unobserved agents’ specific types $\tau_i \sim N(0, \sigma_\tau^2)$ which may influence both the incremental payoff of a friendship and the incremental payoff of smoking. An intuitive example is adding a term $\sum_j |\tau_i - \tau_j| g_{ij}$ to $W(\cdot, \cdot)$ and a term $\rho_\tau \tau_i a_i$ to $V(\cdot)$. In this case, the likelihood has to integrate out $\vec{\tau}$:

$$p(S|\theta) = \int_{\vec{\tau}} \frac{\exp\{\mathcal{P}_\theta(S, \vec{\tau})\}}{\sum_{\hat{S}} \exp\{\mathcal{P}_\theta(\hat{S}, \vec{\tau})\}} \phi(\vec{\tau}) d\vec{\tau}.$$

There are a couple of approaches to discuss the identification of ρ_τ and σ_τ^2 . From a Bayesian perspective, identification obtains as long as the data provides information about the parameters. Even a weakly informative prior can introduce curvature into the posterior density surface that facilitates numerical maximization and the use of MCMC methods. However, the prior distribution is not updated in directions of the parameter space in which the likelihood function is flat (see An and Schorfheide (2007)). From a frequentist perspective, the heuristic identification argument goes as follows. Friends who are far away in observables, must have realizations of the unobservables very close by, that is, small $|\tau_i - \tau_j|$. If in the data those friends tend to make the same smoking choices then it must be the case that $\rho_\tau \sigma_\tau$ is large. However, formalizing this argument is not immediate and may depend on specific parametric assumptions, and so it is left for the future.

4.5. Estimation Results

Table II presents model’s estimates (the posterior means) for four different estimation scenarios: (i) without network data, (ii) with fixed network, (iii) without peer effects and (iv.) the full model. The estimates have been transformed for ease of interpretation to

²¹Most of the parameters are identified in the asymptotic frame where the size of the network grows to infinity (as opposed to the number of networks going to infinity). For example, turning off the externalities ($\phi = 0, \phi_S = 0, \phi_N = 0, q = 0, \psi = 0$) implies that both smoking and friendships are independently distributed so that standard LLNs apply in the single large network asymptotics.

TABLE II
PARAMETER ESTIMATES^a

Parameter	Prior	No Net Data	Exog Net	No PE	Model
<i>Utility of Smoking</i>					
Baseline probability of smoking	0.20 (0.10)	0.12 [0.10, 0.15]	0.17 [0.15, 0.20]	0.21 [0.19, 0.23]	0.18 [0.15, 0.22]
Price × 100	-0.50 (1.00)	-0.17 [-0.36, 0.01]	-0.21 [-0.42, 0.01]	-0.61 [-0.87, -0.37]	-0.24 [-0.48, -0.01]
Mom edu (HS&CO) ^{MP}	-0.05 (0.05)	-0.04 [-0.05, -0.02]	-0.05 [-0.06, -0.03]	-0.05 [-0.07, -0.03]	-0.05 [-0.07, -0.03]
HH smokes ^{MP}	0.10 (0.10)	0.11 [0.08, 0.13]	0.13 [0.11, 0.16]	0.16 [0.13, 0.18]	0.14 [0.11, 0.17]
Grade 9+ ^{MP}	0.20 (0.20)	0.18 [0.14, 0.21]	0.16 [0.12, 0.20]	0.24 [0.20, 0.28]	0.16 [0.11, 0.20]
Blacks ^{MP}	-0.20 (0.20)	-0.30 [-0.34, -0.26]	-0.30 [-0.35, -0.26]	-0.35 [-0.39, -0.31]	-0.31 [-0.37, -0.26]
30% of the school smokes ^{MP}	0.05 (0.10)	0.07 [0.06, 0.08]	0.05 [0.04, 0.07]		0.05 [0.03, 0.08]
<i>Utility of Friendships</i>					
Baseline number of friends	3.00 (2.00)			4.63 [4.09, 5.29]	3.40 [2.88, 3.88]
Different sex ^{MP%}	-0.70 (0.50)			-0.72 [-0.77, -0.67]	-0.72 [-0.77, -0.66]
Different grades ^{MP%}	-0.70 (0.50)			-0.89 [-0.92, -0.86]	-0.89 [-0.92, -0.86]
Different race ^{MP%}	-0.50 (0.50)			-0.33 [-0.58, -0.08]	-0.39 [-0.56, -0.24]
Cost/Economy of scale	0.00 (0.50)			-0.21 [-0.24, -0.18]	-0.22 [-0.24, -0.19]
Triangles ^{MP%}	0.00 (2.00)			1.18 [0.93, 1.45]	1.22 [0.98, 1.45]
$\phi_{\text{smoke}}^{\text{MP}}$	0.05 (0.05)		0.04 [0.04, 0.05]		0.05 [0.04, 0.06]
$\phi_{\text{nosome}}^{\text{MP}}$	0.05 (0.05)		0.03 [0.03, 0.04]		0.04 [0.03, 0.05]

^aNote: All priors are normal distributions with mean and std displayed in the column Prior. The posterior sample contains 10⁵ simulations before discarding the first 20%. Each cell displays the posterior mean and the shortest 90% credible set. MP stands for the estimated marginal probability in percentage points and MP% for estimated marginal probability in percent, relative to the baseline probability.

baseline probabilities, marginal probabilities (MP in ppt) and relative marginal probabilities (MP% in pct).²² It is worth pointing out that the estimate for the price coefficient does not vary much in magnitude (but only in significance). The point estimates in Table II together with the posterior distributions of this parameter in Figure 3 suggest that the largest biases arise when peer effects terms are omitted (column “No PE”) or when the econometrician does not have data on the friendship network (column “No Net Data”). Nevertheless, it is difficult to interpret the magnitudes of these differences nor the magnitudes of the structural estimates altogether. In particular, the reported marginal effects

²²Details about this reparametrization are available in the Online Supplementary Material.

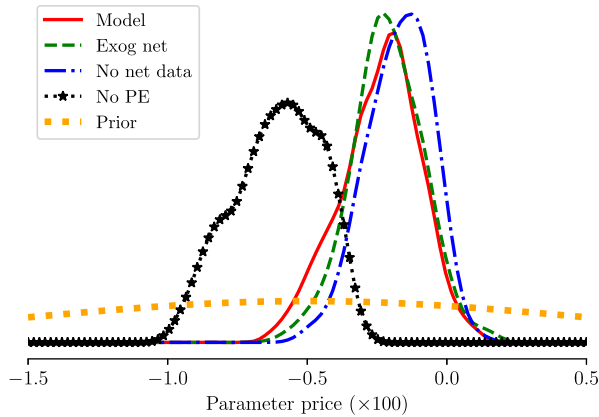


FIGURE 3.—Posterior distribution for the price parameter. *Note:* The hypotheses for equal means between the model’s posterior and each of the other posteriors on the plot are rejected with $p < 0.01$ by t -tests.

represent a first-order response and do not take into account the overall equilibrium adjustments of the entire system.²³

A final point on the estimation results is that the peer effect externalities differ substantially in magnitude between smokers compared to those between nonsmokers. Figure 4 reveals that the former are much stronger than the latter (see footnote 23).

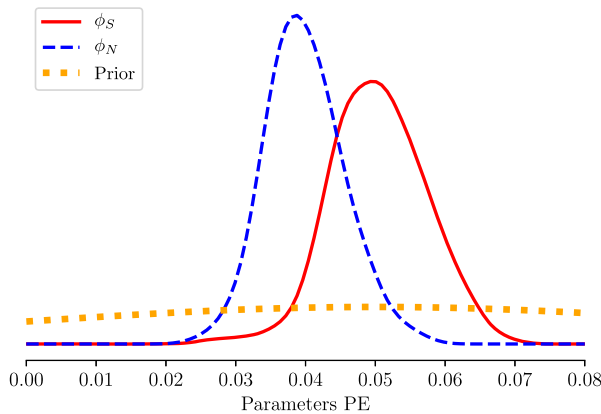


FIGURE 4.—Posterior distribution for the local PE parameters. *Note:* The hypotheses for equal means and equal distributions between the parameters for peer effects among smokers ϕ_S and among nonsmokers ϕ_N are rejected with $p < 0.01$.

²³A related point is that the parameter ϕ_S cannot be interpreted as the effect on the likelihood of smoking from a randomly assigned friend who is a smoker because, in the model, individuals cannot be forced into friendships. Rather, individual’s utility increases with ϕ_S (or ϕ_N) with every instance where her choice to smoke (or not) and her choice of a friend are such that she and this friend of hers both smoke (or not).

TABLE III
THE EFFECT ON SMOKING RATE FROM CHANGES IN THE PRICE OF
TOBACCO^a

Price Increase	Model	Exog Net	No Net Data
20	2.5	2.2	1.3
40	4.7	4.2	2.6
60	6.9	6.1	3.9
80	8.7	7.9	5.1
100	10.3	9.4	6.2
120	11.8	10.9	7.4
140	13.1	12.3	8.4
160	14.3	13.5	9.5

^aNote: The first column shows proposed increases in tobacco prices in cents. The average price of a pack of cigarettes is \$1.67 so that 20 cents is approximately 10%. The second through fourth columns show the predicted increase in the overall smoking (baseline 41%) in ppt from the full model, from the model when the friendship network is fixed, and from the model when no social network data is available (i.e., $\phi_S = \phi_N = 0$).

5. POLICY EXPERIMENTS

5.1. Changes in the Price of Tobacco

The estimated model serves as a numerical prototype for the equilibrium adjustments to various policy interventions. Table III presents simulated increases in tobacco prices ranging from 20 to 160 cents (in the sample tobacco prices average at \$1.67 for a pack) and their effect on the overall tobacco smoking rates for the sample. The table compares the predictions from the full model to those from the model when agents are restricted from adjusting their friendship links and those from the model when data on the friendship network are not available.

As seen in Table III, smoking rates responds to price changes. Comparison between model's predictions with and without friendship adjustments (columns 2 and 3) reveals that the latter underestimates the mean response by around 15%. In addition, the model without friendship choices underestimates the variance of this response as well (see Figure 5). Finally, lack of network data (forcing the restriction $\phi_S = \phi_N = 0$) leads to a bias in the mean response to price changes that is between 50% and 70% of the prediction of the full model.

This analysis suggests that the freedom of breaking friendships and changing smoking behavior induces slightly larger decreases in overall smoking compared to a situation when individuals are held in their existing (fixed) social networks. Figuratively, a price change has two effects on the decision to smoke: the direct effect operates through changing individuals' exogenous decision environment and the indirect/spillover effect operates through changing the peer norm, which then puts additional pressure on the individuals' to follow the change. When comparing the endogenous to fixed network, the direct effect is likely to be stronger in the former environment while the indirect effect is likely to be stronger in the latter environment. This study suggests that quantitatively the direct effect dominates in shaping the overall equilibrium adjustments.²⁴

²⁴It is interesting to relate this finding to the theoretical analysis in Jackson (2018) who argues that variability in individuals' popularity (degree in a social network) leads to biased perceptions for the social norm which in turn leads to higher levels of activities compared to a situation when there is no variability in individuals' popularity. This counterfactual experiment hints to such an amplification mechanism.

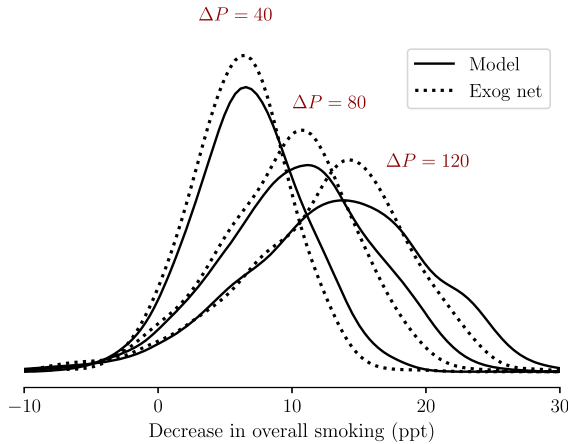


FIGURE 5.—Distribution of the effect on smoking rate from selected price changes. *Note:* In the model with exogenous friendships, the distribution of the predicted smoking rates underestimates both the mean smoking rates (Table III) and the amount of uncertainty associated with the policy intervention (i.e., the variance of the distribution) as compared with the full model. The Appendix provides statistical tests for these differences.

5.2. Changes in the Racial Composition of Schools

Suppose that in a given neighborhood there are two racially segregated schools: “White School” consisting of only white students and “Black School” consisting of only black students. One would expect that the smoking prevalence in White School is much higher compared to Black School because, in the sample, black high students smoke three times less than white high school students. Consider a policy aiming to promote racial desegregation, which prevents schools from enrolling more than x percent of students of the same race. If such policy is in place, will students from different races form friendships and will these friendships systematically impact the overall smoking in one or another direction?

One of the racially balanced schools in the sample is used to evaluate the effect of this policy.²⁵ In particular, the Whites and the Blacks from this school serve as prototypes for the White School and Black School, respectively. To implement the proposed policy, a random set of students from the White School is swapped with a random set of students from the Black School. For example, to simulate the effect of a 70% cap on the same-race students in a school, a swap of 30% is simulated.

Table IV presents the simulation results, which suggest that racial composition affects the overall smoking prevalence. The first column shows the proportion of students being swapped. The second, third, and fourth columns show the simulated smoking prevalence in the White School, Black School, and both, respectively. The table suggests that overall smoking prevalence is lower when schools are racially balanced, thus supporting policies promoting racial integration in the context of fighting high smoking rates.²⁶

²⁵The school has 150 students of which 40% are Whites and 42% are Blacks. It incorporates students from grades 7 to 12. From these, the simulations use students from grades 10 to 12 because older students are more likely to form meaningful friendships and to smoke.

²⁶The statistical power of these predictions is examined in the Online Supplementary Material.

TABLE IV
THE EFFECT ON SMOKING RATES FROM SAME-RACE STUDENTS
CAPS^a

Same Race Cap (%)	School White	School Black	Overall
None	32.9	4.5	18.7
90	29.2	6.7	17.9
80	25.6	9.3	17.4
70	23.6	11.1	17.4
60	18.8	15.0	16.9
50	17.0	16.8	16.9

^aNote: A cap of $x\%$ same-race students is implemented with a swap of $(100 - x)\%$ students. The last column shows the predicted changes in overall smoking under different same-race caps. The policy induces statistically significant changes in the overall smoking as suggested by the statistical tests in Appendix D.

It is important to note that the simulations here offer only suggestive evidence on the role of racial desegregation on the overall prevalence of smoking. There are many factors, for example, the profile of all observables for the entire schools (income, home environment, tobacco price, etc.), that are likely to influence the outcome of desegregation. Unfortunately, the Add Health data does not have much variation in those factors and the counterfactual analysis relies on the only racially balanced school in the data. The author hopes this study to stimulate further research into this question.

5.3. Aggregate Effects of an Antismoking Campaign

The last experiment considers the effects of an antismoking campaign that can prevent with certainty a given number of students from smoking. An example of such intervention is a weekend-long information camp on the health consequences of smoking. Assuming that the camp is very effective in terms of preventing students from smoking but it is too costly to enroll all students in this camp, the question is once the “treated students” come back will their smoking friends follow their example and stop smoking, or will their friends unfriend them and continue smoking?

Table V presents the simulation results with two schools that feature smoking rates at the sample mean. The table suggests that an antismoking campaign may have a large impact on the overall prevalence of smoking, without necessarily being able to directly engage a large part of the student population. In particular, the multiplier factor—the ratio between the actual effect and effect constrained to the treated sub-population—indicated a substantial spillover effects reaching up to the factor of 2. These spillover effects operate through the social network, from those who attended the camp to the rest of the school.

6. CONCLUDING REMARKS

Individuals may respond differently to changes, with some following their friends’ behaviors and others breaking away from their old friends in a search for new friends that will accept their new behaviors. The choice of a friend fundamentally differs from the choice of behaviors and, moreover, modeling both choices simultaneously generates complex mathematical structures. In the proposed equilibrium, players internalize the need

TABLE V
SPILLOVERS^a

Campaign (%)	Smoking	Predicted Effect Proportional	Actual Effect	Multiplier
–	42.1	–	–	
3	39.6	1.3	2.6	2.0
5	38.2	2.1	3.9	1.9
10	34.6	4.2	7.5	1.8
20	28.7	8.4	13.4	1.6
30	23.5	12.6	18.6	1.5
50	15.1	21.1	27.0	1.3

^aNote: The first column lists the various treatment rates (proportion of treated). The second and third columns display the smoking rate and the change in smoking rate, respectively, if only treated were to stop smoking (i.e., a baseline without peer effects). The fourth column reports the overall equilibrium effect. Finally, the last column displays the multiplier computed as the ratio between columns 3 and 4. Note that the treatment is random and does not target smokers. The policy is simulated 10³ times, where each time a new random draw of attendees is being considered.

for consensus in forming friendships and choose their optimal strategies on subsets of k players, that is, considering the optimality of only a few friendships at a time—a form of bounded rationality. The k -player consensual dynamic—an adaptive dynamic where players are constrained to forming friendships with consent—delivers a probabilistic ranking of the proposed equilibria, and via a varying k , facilitates the implementation of the model.

The model is used to empirically study adolescents’ smoking and friendships selection. The estimation results suggest that: (a) peer effect complementarities between smokers are substantially stronger than those between nonsmokers, and (b) lack of social network data biases substantially the estimates for the marginal effect of tobacco price on adolescent smoking. Counterfactual analysis with the estimated model suggests that: (a) the response of the friendship network to changes in tobacco price amplifies the intended effect of price changes on smoking, (b) racial desegregation of high schools decreases the overall smoking prevalence, and (c) the equilibrium effects from small scale policies targeting individuals’ smoking choices are roughly double compared to the scale of these policies.

Overall this paper formulates an avenue to study the complementarities and coordination in live social networks, that is, social networks that adapt to the behaviors of individuals. The literature has just started to understand the forces present in these environments (e.g., see Jackson (2018)) while the empirical investigation of many hypotheses remains for the future (e.g., see Carrell, Sacerdote, and West (2013), Graham, Imbens, and Ridder (2014)).

APPENDIX: PROOFS

PROOF OF PROPOSITION 1 (ON P. 1186): The trivial solution is always feasible w.r.t. the stability constraints (5) and so the domain of maximization (4) is nonempty. Further, the hypothesis implies the existence of a nontrivial solution. Note that $\Delta_{g_{ij}} u_i(\cdot) = \Delta_{g_{ij}} u_j(\cdot)$. This property of the preferences implies that the unconstrained maximum in (4) is feasible w.r.t. the stability constraints (5). That is, for any i and $I_k = \{i\} \cup \{i_1, \dots, i_{k-1}\}$ the solution

of individual’s decision problem (4)–(5) is simply

$$\operatorname{argmax}_{\substack{a_i, g_{ij} \\ j \in I_k \setminus i}} \mathcal{P}(S). \tag{4}$$

This completes the proof of part one.

For part two, the first step is to extend the property $\Delta_{g_{ij}} u_i(\cdot) = \Delta_{g_{ij}} u_j(\cdot)$ to a deeper property of the preferences namely that the preferences of all players can be expressed by a single potential function.²⁷ Indeed, consider $\mathcal{P} : \mathbf{S} \times \mathbf{X}_n \rightarrow \mathbb{R}$:

$$\begin{aligned} \mathcal{P}(S, X) &= \sum_i a_i v(X_i) + \frac{1}{2} \sum_{i,j} g_{ij} w(X_i, X_j) \\ &+ \frac{1}{2} \phi \sum_{i,j:i \neq j} a_i a_j + \frac{1}{2} \phi_S \sum_{i,j} g_{ij} a_i a_j + \frac{1}{2} \phi_N \sum_{i,j} g_{ij} (1 - a_i)(1 - a_j) \\ &+ \frac{1}{6} \sum_{i,j,k} q(X_i, X_j, X_k) g_{ij} g_{jk} g_{ki}, \end{aligned}$$

where $i \neq j$ is dropped from the summation ranges where possible because the convention that g_{ii} is defined and equals to 0 for all i so that $\sum_{i,j:i \neq j} g_{ij} = \sum_{i,j} g_{ij}$. To show that \mathcal{P} is potential, it is sufficient to verify that (using Assumption 1):

$$\begin{aligned} \Delta_{a_i} u_i(\cdot) &= v_i + \phi \sum_{j \neq i} a_j + \phi_S \sum_{j \neq i} g_{ij} a_j - \phi_N \sum_{j \neq i} g_{ij} (1 - a_j) \\ &= \Delta_{a_i} \mathcal{P}(\cdot), \\ \Delta_{g_{ij}} u_i(\cdot) &= w_{ij} + q(X_i, X_j, X_k) \sum_k g_{ik} g_{jk} \\ &\quad - \psi(d_i + d_j) + \phi_S a_i a_j + \phi_N (1 - a_i)(1 - a_j) \\ &= \Delta_{g_{ij}} \mathcal{P}(\cdot). \end{aligned}$$

Next, fix $k \in \{2, \dots, n\}$ and consider the following adaptive dynamic on \mathbf{S} . Every period draw at random i and I_k (from the uniform distributions over their respective domains), and let i choose in her argmax (4). For this dynamic, the value of the potential is non-decreasing so, invoking a submartingale convergence argument, the potential converges. Unless two states have the same potential (generically false), this implies that the state converges to a particular network which is, of course, a NE k S network. This same technology appears in the proof of Proposition 3. Q.E.D.

As it will be useful later on, Proposition 5 states the equivalence between individual’s decision problem (4)–(5) and the unconstrained maximization (4). The proof of the if direction follows closely that of the only if direction above, and is omitted.

²⁷The existence of potential implies $\Delta_{g_{ij}} u_i(\cdot) = \Delta_{g_{ij}} u_j(\cdot)$ but the converse is not true.

PROPOSITION 5: Fix $k \in \{2, \dots, n\}$. S^* is a NEkS network iff $\forall i, I_k = \{i\} \cup \{i_1, \dots, i_{k-1}\}$,

$$(a_i^*, g_{ij}^*)_{j \in I_k \setminus i} \in \operatorname{argmax}_{\substack{a_i, g_{ij} \\ j \in I_k \setminus i}} \mathcal{P}((a_i, g_{ij})_{j \in I_k \setminus i}; S_{-(a_i, g_{ij})_{j \in I_k \setminus i}}^*).$$

PROOF OF PROPOSITION 2 (ON P. 1186): For $k = 2$, definition 1 directly implies that a NEkS network is pairwise stable. Note that this observation is independent of the particular payoff structure here.

Let $k = n$. That a NEkS network is pairwise stable follows from part 3 of this proposition (demonstrated next). To see that a NEkS network S^* is a Nash network, consider the following strategies in a normal form link-announcement game (given the equilibrium behavior \vec{a}^*): each player announces his NEkS links. Proceeding by contradiction, for if a player has a profitable deviation then it would be possible to construct (appending a_i^*) an $S_{(i)}$ which she prefers to her NEkS play $S_{(i)}^*$. Therefore, $S_{(i)}^* \notin \operatorname{argmax}_{j \in I_k \setminus i}^{a_i, g_{ij}} u_i(S) = \operatorname{argmax}_{j \in I_k \setminus i}^{a_i, g_{ij}} \mathcal{P}(S)$, which contradicts Proposition 5.

Finally, the characterization from Proposition 5 directly implies part 3. In particular, if $k' < k$, $I_{k'} \subset I_k$ and $(a_i^*, g_{ij}^*)_{j \in I_k \setminus i} \in \operatorname{argmax}_{j \in I_k \setminus i}^{a_i, g_{ij}} \mathcal{P}((a_i, g_{ij})_{j \in I_k \setminus i}; S_{-(a_i, g_{ij})_{j \in I_k \setminus i}}^*)$ then $(a_i^*, g_{ij}^*)_{j \in I_{k'} \setminus i} \in \operatorname{argmax}_{j \in I_{k'} \setminus i}^{a_i, g_{ij}} \mathcal{P}((a_i, g_{ij})_{j \in I_{k'} \setminus i}; S_{-(a_i, g_{ij})_{j \in I_{k'} \setminus i}}^*)$. Q.E.D.

PROOF OF PROPOSITION 3 (ON P. 1188): That any NEkS network is absorbing for the k CD follows from Definition 1. The second part follows from observing that \mathcal{P}_t is a submartingale, that is, $E[\mathcal{P}_{t+1}|S_t] \geq \mathcal{P}_t$, so that $\{\mathcal{P}_t\}$ converges almost surely. Because the network size is finite it follows that $\{\mathcal{P}_t\}$ is constant for large t and, generically, the same holds for S_t , that is, $S_t = S^*$ for large enough t . Because of Assumption 2 (any meeting is possible), this can happen only if S^* is a NEkS network. Q.E.D.

PROOF OF THEOREM 1 (P. 1189): The first part follows from standard results on convergence of Markov chains. In particular, k -CDs with random utility induce a finite state Markov chain which, with Assumptions 2 and 3, is irreducible, positive recurrent, and aperiodic. This is sufficient to obtain the conclusion of part 1.

For the second part, it is enough to show that

$$\Pr(S'|S; k) \exp\{\mathcal{P}(S)\} = \Pr(S|S'; k) \exp\{\mathcal{P}(S')\}, \tag{11}$$

where $\Pr(S'|S; k)$ is the one step transition probability for moving from S to S' .

There are two cases to consider: $\Pr(S'|S; k) = 0$ and $\Pr(S'|S; k) > 0$. Suppose that $\Pr(S'|S; k) = 0$. In this case, the hypothesis guarantees that S and S' differ in the play of more than one player. Therefore, $\Pr(S|S'; k) = 0$ and, trivially, (11) holds.

Consider the case $\Pr(S'|S; k) > 0$. Let $\mathbf{M}_{S'|S; k}$ be the set of all possible meetings that can result in transitioning from S to S' . Note that $\mathbf{M}_{S'|S; k}$ is empty for some triples (S, S', k) . Also, $\Pr(S'|S; k) > 0$ implies $\mathbf{M}_{S'|S; k} \neq \emptyset$.

Let us pause with an example of this notation. Given the triple (S, S', k) ,

$$\Pr(S'|S; k) = \sum_{\mu \in \mathbf{M}_{S'|S; k}} \Pr(\mu) \frac{\exp\{u_i(S')\}}{\sum_{\hat{S} \in \mathbf{N}_k(\mu, S)} \exp\{u_i(\hat{S})\}},$$

where $\mathbf{N}_k(\mu, S)$ stands for all states that can result from state S and meeting μ . Here, the well known expression for the logit choice probabilities (once the meeting is fixed) follows from Assumption 4 on the distribution of the error term. Suppose that S and S' agree on all $\{g_{ij}\}_{i \neq j}$ but differ in a_i for some i , say $S = (a_i = 0, S_{-a_i})$ and $S' = (a'_i = 1, S_{-a_i})$. For such S and S' , $\mathbf{M}_{S'|S;k}$ is the set of all possible meeting tuples (i, I_{k-1}) where player i meets different $\{i_1, \dots, i_{k-1}\}$, and the size of $\mathbf{M}_{S'|S;k}$ is $\binom{n-1}{k-1}$. Further, assume that all meetings are equally likely and that individuals are indifferent to all outcomes (i.e., as in the hypothesis of Theorem 2, $u_i(S, X) = 0$ for all i , $S \in \mathbf{S}$ and $X \in \mathbf{X}$). With these assumptions, $\Pr(\mu) = \frac{1}{n} \frac{1}{\binom{n-1}{k-1}}$ and $\frac{\exp\{u_i(S')\}}{\sum_{\hat{S} \in \mathbf{N}_k(\mu, S)} \exp\{u_i(\hat{S})\}} = \frac{1}{2^k}$, and the example is complete with

$$\Pr(S'|S; k) = \binom{n-1}{k-1} \frac{1}{n \binom{n-1}{k-1}} \frac{1}{2^k} = \frac{1}{n2^k}.$$

Recall that $\mathbf{N}_k(S, \mu) \subset \mathbf{S}$ denotes the set of all possible states that can result from the meeting μ following a state S . The proof of (11) in the case of $\Pr(S'|S; k) > 0$ follows from the following observations.²⁸

- LEMMA 1: For all k, S, S' , and $\mu = (i, I_{k-1})$:
- (i) $\mathbf{M}_{S'|S;k} = \mathbf{M}_{S|S';k}$ for all $S, S' \in \mathbf{S}_n$;
 - (ii) $S' \in \mathbf{N}_k(\mu, S)$ iff $S \in \mathbf{N}_k(\mu, S')$;
 - (iii) If $S' \in \mathbf{N}_k(\mu, S)$ then $\mathbf{N}_k(\mu, S) = \mathbf{N}_k(\mu, S')$.

Part (i) asserts that each meeting that can result in transitioning from S to S' may result in transitioning from S' to S as well (provided the starting state were S'). Part (ii) restates this observation in terms of the neighborhoods of S and S' given a meeting μ . Finally, part (iii) notes that if a meeting μ could result in S transiting to S' , then the set of all feasible states following μ and S coincides with the set of all feasible states following μ and S' .

From Lemma 1, the one step transition probability can be written as

$$\begin{aligned} \mathcal{P}(S) \Pr(S'|S; k) &= \mathcal{P}(S) \sum_{\mu \in \mathbf{M}_{S'|S;k}} \Pr(\mu) \frac{\exp\{u_i(S')\}}{\sum_{\hat{S} \in \mathbf{N}_k(\mu, S)} \exp\{u_i(\hat{S})\}} \\ &= \mathcal{P}(S) \sum_{\mu \in \mathbf{M}_{S|S';k}} \Pr(\mu) \frac{\exp\{\mathcal{P}(S')\}}{\sum_{\hat{S} \in \mathbf{N}_k(\mu, S')} \exp\{\mathcal{P}(\hat{S})\}} \\ &= \mathcal{P}(S') \Pr(S|S'; k). \end{aligned}$$

The expression for logit choice probabilities follows from Assumption 4 on the distribution of the error term. In that expression, substituting with $\mathcal{P}()$ for $u_i()$ follows from Assumption 1 which guarantees the existence of a potential. Q.E.D.

²⁸The proof of Lemma 1 involves basic reasoning and is omitted. The challenging part is to state and interpret the lemma. Formal proof available upon request.

PROOF OF THEOREM 2 (P. 1189): Because there is no natural ordering of \mathbf{S} , use functions as opposed to vectors in the eigenproblem. For $I \subset \{(i, j) : i \geq j\}$, define $e_I : \mathbf{S} \rightarrow \mathbb{R}$ as

$$e_I(S) = \prod_{i \neq j \in I} (-1)^{s_{ij}} \prod_{i=j \in I} (-1)^{a_i}$$

with $e_\emptyset(S) = 1$ for all S . Next, define

$$\lambda_{k,I} = \frac{\sum_{i \in \{(i,i) \notin I\}} \binom{n-1-|I_i|}{k-1}}{n \binom{n-1}{k-1}}$$

where $I_i = \{j : (i, j) \in I, i \neq j\}$.

LEMMA 2: There are $2^{n(n+1)/2}$ pairs of $(\lambda_{k,I}, e_{k,I})$ such that:

- (i) $\sum_S e_{k,I}(S)e_{k,I'}(S) = 0$ if $I \neq I'$ and $\sum_S e_{k,I}(S)e_{k,I}(S) = 2^{n(n+1)/2}$;
- (ii) $\sum_{S'} \Pr(S'|S; k)e_I(S') = \lambda_{k,I}e_I(S)$ for all $S \in \mathbf{S}$.

To demonstrate part (i), suppose that $I \neq I'$ and let $\tilde{I} = I \setminus I' \cup I' \setminus I \neq \emptyset$. Now note that $\sum_S e_{k,I}(S)e_{k,I'}(S) = \sum_S e_{k,\tilde{I}}(S) = 0$. The rest is trivial to verify. For part (ii), write

$$\sum_{S'} \Pr(S'|S)e_I(S') = \sum_{S'} \sum_{\mu} \Pr(\mu) \Pr(S'|S, \mu)e_I(S') \tag{12}$$

$$= \sum_{S'} \sum_{\mu \in \{\mu \cap I = \emptyset\}} \Pr(\mu) \Pr(S'|S, \mu)e_I(S') \tag{13}$$

$$+ \sum_{S'} \sum_{\mu \in \{\mu \cap I \neq \emptyset\}} \Pr(\mu) \Pr(S'|S, \mu)e_I(S') \tag{14}$$

$$= \sum_{S'} \sum_{\mu \in \{\mu \cap I = \emptyset\}} \Pr(\mu) \Pr(S'|S, \mu)e_I(S'). \tag{15}$$

Terms (14) vanish because μ is such that $\{(i, i), (i, i_1), \dots, (i, i_{k-1})\} \cap I \neq \emptyset$. Indeed, for $\sum_{S' \in \mathbf{N}_k(S, \mu)} \Pr(S'|S, \mu)e_I(S')$ in half of the 2^k terms $e_I(S') = e_I(S)$ while for the other half $e_I(S') = -e_I(S)$, implying $\sum_{\mu \in \{\mu \cap I \neq \emptyset\}} \sum_{S' \in \mathbf{N}_k(S, \mu)} \Pr(S'|S, \mu)e_I(S') = 0$.

In summation (15), $e_I(S) = e_I(S')$ for all $S' \in \mathbf{N}_k(S, \mu)$ because μ is such that $\{(i, i), (i, i_1), \dots, (i, i_{k-1})\} \cap I = \emptyset$. Also, $\Pr(\mu) = \frac{1}{n \binom{n-1}{k-1}}$ and $\sum_{S'} \Pr(S'|S, \mu) = 1$. Therefore, continuing with summation (15),

$$\begin{aligned} \sum_{S'} \Pr(S'|S)e_I(S') &= e_I(S) \Pr(\mu) \sum_{S'} \sum_{\mu \in \{\mu \cap I = \emptyset\}} \Pr(S'|S, \mu) \\ &= e_I(S) \frac{1}{n \binom{n-1}{k-1}} \sum_{\mu \in \{\mu \cap I = \emptyset\}} \sum_{S'} \Pr(S'|S, \mu) \\ &= e_I(S) \frac{1}{n \binom{n-1}{k-1}} \sum_{i \in \{(i,i) \notin I\}} \binom{n-1-|I_i|}{k-1}. \end{aligned}$$

This completes the proof of Lemma 2. To complete the proof of the theorem, note that $\lambda_{k,I}$ are decreasing in $|I|$ and the second largest $\lambda_{k,I}$ is for $I = \{(i, j)\}$ with $i \neq j$. *Q.E.D.*

PROOF OF THEOREM 3 (P. 1191): The proof follows immediately from the expression for the stationary distribution obtained in Theorem 1 and Proposition 5. *Q.E.D.*

PROOF OF PROPOSITION 4 (P. 1192): For fixed $S, S' \in \mathbf{S}$, let $\mathbf{K}_{S'|S} \subset \{2, 3, \dots, n\}$ be the set of all meeting sizes k consistent with the possibility of transitioning from S to S' of a k CD with random utility. Recall that, for a fixed k , $\mathbf{M}_{S'|S;k}$ is the set of all meetings of size k consistent with the possibility of transitioning from S to S' . The proof follows from Lemma 1, together with the observation that $\mathbf{K}_{S'|S} = \mathbf{K}_{S|S'}$. Indeed, the unconditional proposal Q from the algorithm in Table I can be written as

$$\begin{aligned} Q(S'|S) &= \sum_{k \in \mathbf{K}_{S'|S}} p_k(k) \sum_{\mu \in \mathbf{M}_{S'|S;k}} \Pr(\mu) q_\mu(S'|S; \mu) \\ &= \sum_{k \in \mathbf{K}_{S|S'}} p_k(k) \sum_{\mu \in \mathbf{M}_{S|S';k}} \Pr(\mu) q_\mu(S|S'; \mu) \\ &= Q(S|S'). \end{aligned} \quad \text{Q.E.D.}$$

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